Holography implementation for LOFAR

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1 Averaging and calibrating the visibilities

We start from the input data, the visibilities $V_i(t, R, b, \nu)$, containing the cross correlation of the station we aim to calibrate with all the reference stations, as a function of time t, reference station R, beamlet b, frequency ν. Each element contains the four products of the polarisations from the two stations as a coherency matrix

$$\begin{bmatrix} XX^* & XY^* \\ YX^* & YY^* \end{bmatrix}$$

Step 1: Average over time. Average V_i over M time samples. The averaging in this step should be small (1-10 seconds) as the ionospheric and other time variable effects are only removed in the next step.

$$V_a(T_l, R, b, \nu) = \frac{1}{M} \sum_{m=l\cdot M}^{(l+1)\cdot M} V_i(t_m, R, b, \nu) \tag{1}$$

With new time $T_l = \frac{t_{l\cdot M} + t_{(l+1)\cdot M}}{2}$, and the summation is performed in each polarization (XX^*, XY^*, YX^*, YY^*).

There is also averaging in time in the pre-processing pipeline, but for commissioning purposes, it's also required in this analysis step.

Step 2: Right multiply with the inverse visibility V_a at the reference beamlet rb for each timestep.

$$V_C(T_I, R, b, \nu) = V_a(T_l, R, b, \nu) V_a(T_l, R, rb, \nu)^{-1} \tag{2}$$

For a 2 x 2 matrix the inverse is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Step 3: Average again over time, now over the full duration. We could select one polarisation P here to analyse.

$$V_R(R,b,\nu,P) = \frac{1}{M}\sum_{m=0}^{M} V_C(T_m,R,b,\nu,P) \tag{3}$$

Step 4: Average over the reference stations, using weight w_i

$$V(b,\nu,P) = \frac{1}{w(b,\nu,P)}\sum_{i=0}^{N_R} w_i(b,\nu,P)V_R(R_i,b,\nu,P) \tag{4}$$

with N_R the total number of stations and $w(b,\nu,P) = \sum_i w_i(b,\nu,P)$. A choice for the weights is $w_i = 1/\sigma^2_{V_R}(R_i,b,\nu,P)$, but we also want to be able to specify other weights, that for example only change per station and not per beam.

2 Error propagation

We'll need to do proper error propagation. We'll start with determining the uncertainty in step 3, if enough time samples are available.

Step 3: The uncertainty σ_{V_R} on V_R is the standard deviation with degrees of freedom $M-1$. The standard deviation is determined for the real (Re) and imaginary (Im) part separately.

$$\sigma_{V_R}(R,b,\nu,P) = \sqrt{\frac{1}{M-1}\sum_{m=0}^{M}(Re[V_C(T_m,R,P)-V_R(R,P)])^2} \tag{5}$$

$$+ \; i\sqrt{\frac{1}{M-1}\sum_{m=0}^{M}(Im[V_C(T_m,R,P)-V_R(R,P)])^2} \tag{6}$$

where on the right side dependencies on b,ν are not shown.

Step 4: We are averaging using weights w_i, thus the uncertainties should also be weighted with these values. Given $< V >= \frac{1}{w}\sum_i w_i V_i$

$$\sigma^2_{<V>} = \sum \left(\frac{d<V>}{dV_i}\right)^2 \sigma^2_{V_i} = \frac{1}{w^2}\sum_i w_i^2 \sigma^2_{V_i} \tag{7}$$

Thus

$$\sigma_V(b,\nu,P)^2 = \frac{1}{w^2}\sum_{i=0}^{N_R} w_i^2 \sigma^2_{V_R}(R_i,b,\nu,P) \tag{8}$$

In case $w_i = \frac{1}{\sigma^2_{V_R}(R_i,b,\nu,P)}$ this becomes

$$\sigma_V(b,\nu,P)^2 = \frac{1}{w(b,\nu,P)} \tag{9}$$

but in general the weights may be different.

3 Solving for the gains

Step 5: Solve the linear equation for each frequency ν and polarization P

$$V(b,\nu,P) = M(\nu,b,A)\cdot G(\nu,P,A) \tag{10}$$

with $G(\nu,P,A)$ the gain of antenna A at frequency ν for polarization P, $M_{b,A} = \frac{1}{N_A}e^{-2\pi i\nu(l_b x_A + m_b y_A)/c}$ with N_A the total number of antennas.

As M is not a square matrix, we'll need to find the best solution to the following equation:

$$G = (M^T M)^{-1} M^T V \tag{11}$$

As there is an uncertainty in the measumerment of V, the solution is not exact. One way to solve this, is with a least squares approach. This is implemented in the numpy.linalg.lstsq function.

We may trust some visibilities more than others. Therefore we also want to be able to solve a weighted equation of this form. This is the solution to:

$$G = (M^T W M)^{-1} M^T W V \tag{12}$$

where W are the squares of the weights given to V. The default value for W is a diagonal matrix of the form:

$$W_{bb} = \frac{1}{\sigma^2_V(b,\nu,P)} \tag{13}$$

This would be the solution to the equation:

$$wV(b, \nu, P) = wM(\nu, b, A) \cdot G(\nu, P, A) \tag{14}$$

where $w = \sqrt{W}$. It should also be possible to give a costum w or W function in the (re-)analysis.

4 Interpolating the gains

The gain solution itself is the gain at each subband. We assume the gain is of the form:

$$G(\nu) = (a + b\nu)e^{2\pi i\tau\nu+\phi_0} \tag{15}$$

This is solved for every antenna A and polarization P for parameters a, b, τ, ϕ_0 by using a Monte Carlo procedure. This also gives an estimate on σ_G. We should also be able to estimate σ_G from the calculations above, but this method still needs to be derived. We so far have picked a reference antenna first (typically antenna 0) and divided the gains by the value for that antenna. Some further tests need to verify if that is actually the best way to do this.